

ON A CLASS OF C\*-PREDUALS OF  $l_1$ 

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**Abstract** As it is well known, the Banach space  $l_1$  of absolutely summable (complex) sequences endowed with the  $\|\cdot\|_1$  norm is not *unique predual*. This means that there are many different (*i.e.* non isometrically isomorphic) Banach spaces  $X$  such that  $X^* \cong l_1$ .

The present note is aimed to point out a simple class of C\*-preduals of  $l_1$ : namely the spaces  $C_\tau(\mathbb{N})$  of continuous functions  $f : \mathbb{N} \rightarrow \mathbb{C}$ , where the set of natural numbers  $\mathbb{N}$  is equipped with a compact Hausdorff topology  $\mathcal{T}$ .

To be more concrete, we shall explicitly describe a countable collection  $\{\mathcal{T}_n\}$  of such topologies.

Finally, we also provide an abstract characterization of the previous preduals as closed subspaces  $M \subset l^\infty$  rich of positive elements.

As commonly used in the literature, we shall denote by  $l_1$  the (complex) Banach space of absolutely summable sequences, given of the norm  $\|\cdot\|_1$  defined by  $\|a\|_1 \doteq \sum_{i=1}^{\infty} |a_i|$  for each  $a \in l_1$ .

It is a very well known fact that  $l_1$  is a conjugate Banach space, that is there exists at least a Banach space  $X$ , such that  $X^* \cong l_1$  (isometric isomorphism). Such a space is usually named a *predual*. The most famous predual of  $l_1$  is probably represented by the space  $c_0$  of those (complex) sequences converging to 0, endowed of the *sup*-norm. In this case, the isometric isomorphism  $c_0^* \cong l_1$  is the map  $\Psi : l_1 \rightarrow c_0^*$  given by  $\langle \Psi(y), x \rangle \doteq \sum_{i=1}^{\infty} y_i x_i$  for every  $x \in c_0$  and  $y \in l_1$ .

In spite of its simple definition,  $l_1$  is a rather pathological<sup>1</sup> Banach space: for instance the predual is not unique; there is in fact a plenty of (non isomorphic) preduals of  $l_1$ . Some of these are quite "irregular": Y. Benyamini and J. Lindenstrauss [4] proved in 1972 that there is a predual of  $l_1$  that is not (topologically) complemented in any  $C(K)$ -space,  $K$  being any compact Hausdorff topological space.

On the other hand, the present paper is aimed to discuss a very nice class of C\*-preduals of  $l_1$ . In this spirit, the first thing that should be noticed is the following:

**Proposition 1.** *If  $\mathcal{T}$  is a compact Hausdorff topology on the set of natural numbers  $\mathbb{N}$ , one has  $C_\tau(\mathbb{N})^* \cong l_1$ .*

*Proof.* It is possible to prove the statement by using the Riesz-Markov theorem. Here we perform a proof based on the characterization of separable conjugate

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<sup>1</sup>The weak topology of  $l_1$  is not well behaved: every weakly convergent sequence is indeed norm-convergent, although the weak topology is strictly weaker than the norm topology.

spaces given in [5]. To this aim, we only have to check that  $C_\tau(\mathbb{N}) \subset l^\infty$  is a closed, norm-attaining and 1-norming subspace.

$C_\tau(\mathbb{N})$  is closed in  $l^\infty$  as a complete subspace. It is norm-attaining (when it is thought as subspace of bounded linear functionals on  $l_1$ ) thanks to Weierstrass' theorem, since  $(\mathbb{N}, \mathcal{T})$  is a compact space by assumption.

If  $y \in l_1$  and  $\varepsilon > 0$ , there is  $n \in \mathbb{N}$  such that  $\|y\|_1 \leq \sum_{i=1}^n |y_i| + \varepsilon$ . Let  $\theta_i \in \mathbb{R}$  such that  $y_i = |y_i|e^{i\theta_i}$  for each  $i = 1, 2, \dots, n$ . The subset  $C_n \doteq \{1, 2, \dots, n\} \subset \mathbb{N}$  is closed (and discrete), hence the function  $f : C_n \rightarrow \mathbb{C}$  given by  $f(i) = e^{-i\theta_i}$  for each  $i \in C_n$  is continuous and  $\|f\|_\infty = 1$ . Since  $(\mathbb{N}, \mathcal{T})$  is a compact Hausdorff space, it is a normal topological space, so Tietze extension theorem applies to get a function  $g \in C_\tau(\mathbb{N})$  such that  $\|g\|_\infty = 1$  and  $g(i) = e^{-i\theta_i}$  for each  $i \in 1, 2, \dots, n$ .

We have  $|\langle g, y \rangle| = |\sum_{i=1}^\infty g(i)y_i| \geq \sum_{i=1}^n |y_i| - \varepsilon \geq \|y\|_1 - 2\varepsilon$ . The last inequality easily implies that

$$\sup_{g \in C_\tau(\mathbb{N})_1} |\langle g, y \rangle| = \|y\|_1$$

that is  $C_\tau(\mathbb{N}) \subset l^\infty$  is a 1-norming subspace. This ends the proof.  $\square$

The previous proposition immediately leads to the following corollary in point-set topology:

**Corollary 2.** *Every compact Hausdorff topology on the set of natural numbers  $\mathbb{N}$  is metrizable.*

*Proof.* Let  $\mathcal{T}$  be such a topology. We have  $C_\tau(\mathbb{N})^* \cong l_1$ , hence  $C_\tau(\mathbb{N})$  is a separable Banach space, as a predual of the separable Banach space  $l_1$ , so that  $(\mathbb{N}, \mathcal{T})$  is metrizable.  $\square$

**Note 3.** As far as I know, a simple proof of the corollary quoted above does not seem available in the general setting of point-set topology, since it is not apparent that a compact Hausdorff topology on  $\mathbb{N}$  is automatically second countable.

On the other hand, non first countable topologies on  $\mathbb{N}$  are known: *Appert* topology, for instance, provides an elegant example of such a space. For the reader's convenience, we recall here that Appert's topology on  $\mathbb{N}$  is defined as follows: a subset  $A \subset \mathbb{N}$  is open if  $1 \notin A$  or (when  $1 \in A$ ) if

$$\lim_{n \rightarrow \infty} \frac{N(n, A)}{n} = 1$$

where  $N(n, A) \doteq |\{k \in A : k \leq n\}|$ .<sup>2</sup> Appert space is Lindelöf, separable but it is not first countable, since 1 does not have a countable basis of neighborhoods. For more details, we refer the interested reader to [6] or directly to the original paper by Appert [1].

Here below we shall describe explicitly a countable collection of compact Hausdorff topologies on  $\mathbb{N}$ . Before introducing the announced topologies, one should mention that every set  $X$  can be endowed with a compact Hausdorff

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<sup>2</sup> $|X|$  is the cardinality of any set  $X$ .

topology, by virtue of a straightforward application of the Axiom of Choice<sup>3</sup>. Now let  $n \in \mathbb{N}$  be a fixed natural number. Given any  $k \in \{1, 2, \dots, n\}$ , we define the sets  $A_{k,l} \doteq \{k, mn + k : m \geq l\}$ . The sets  $A_{k,l}$  allow us to define a topology  $\mathcal{T}_n$ , whose basis  $\mathcal{B}_n$  is given by the subset  $B \subset \mathbb{N}$  of the form  $A_{k,l}$  if  $k \in B$  for some  $k \in \{1, 2, \dots, n\}$ , otherwise we do not put any restriction, namely if  $\{1, 2, \dots, n\} \cap B = \emptyset$  then  $B$  is allowed to be any subset of the natural numbers. Since  $A_{k,l} \cap A_{k,h} = A_{k,l \vee h}$ <sup>4</sup> and  $A_{k,l} \cap A_{k',h} = \emptyset$  when  $k, k' \in \{1, 2, \dots, n\}$  are different,  $\mathcal{B}_n$  is really a basis. It is a straightforward verification to check that  $\mathcal{T}_n$  is a compact Hausdorff topology; the notion of convergence inherited by this topology is clearly the following:

a sequence  $\{n_m : m \in \mathbb{N}\}$  of integers converges to  $k \in \{1, 2, \dots, n\}$  iff  $n_m$  is eventually in a set  $A_{k,l}$ , while converges to  $k > n$  iff it is eventually equal to  $k$ . In the topology  $\mathcal{T}_n$  the set  $\{k : k \leq n\}$  is composed by non isolated points, while all the integers  $k > n$  are isolated. In some sense, topologies  $\mathcal{T}_n$  are as best as possible among compact Hausdorff ones, since it is a straightforward application of *Baire* category theorem that a compact Hausdorff topology on  $\mathbb{N}$  cannot have an infinite set of accumulation points<sup>5</sup>.

However, what is more important here is that a simple argument can be performed to prove that the topologies  $\mathcal{T}_n$  are not homeomorphic:

**Proposition 4.** *With the notations above, if  $n \neq m$  the topological spaces  $(\mathbb{N}, \mathcal{T}_n)$  and  $(\mathbb{N}, \mathcal{T}_m)$  are not homeomorphic.*

*Proof.* Let us suppose that  $m > n$  and let  $\Phi : (\mathbb{N}, \mathcal{T}_m) \rightarrow (\mathbb{N}, \mathcal{T}_n)$  be a continuous injective map. If  $k \in \{1, 2, \dots, m\}$ , we can consider a sequence  $\{n_l\}$  converging to  $k$ . The sequence  $\{\Phi(n_l)\}$  converges to  $\Phi(k)$  thanks to the continuity of  $\Phi$ . Since  $\{n_l\}$  is not constant and  $\Phi$  is an injection  $\Phi(k)$  is forced to be a natural number belonging to the subset  $\{1, 2, \dots, n\}$ , against the injectivity of  $\Phi$ .  $\square$

Let us denote by  $X_n$  the Banach space  $C_{\tau_n}(\mathbb{N})$ . Clearly we have  $X_n^* \cong l_1$  and

**Proposition 5.** *If  $n \neq m$  the Banach space  $X_n$  and  $X_m$  are  $l_1$ -preduals, which are not isometrically isomorphic.*

*Proof.* If they were isometrically isomorphic, the topological space  $(\mathbb{N}, \mathcal{T}_n)$  and  $(\mathbb{N}, \mathcal{T}_m)$  should be homeomorphic according to the classical Banach-Stone theorem.  $\square$

The remaining part of the present paper is devoted to provide an intrinsic characterization of the spaces  $C_\tau(\mathbb{N})$  as suitable subspaces of  $l^\infty$ . To this aim, one probably has to remind that any predual  $M$  of a conjugate spaces  $X$  should

<sup>3</sup>The discrete topology  $\mathcal{P}(X)$  on  $X$  is locally compact and Hausdorff. The Alexandroff compactification  $\hat{X}$  of  $X$  is compact and Hausdorff; moreover, if  $X$  is an infinite set, there is a bijection  $\Phi : X \rightarrow \hat{X}$ . We can use  $\Phi$  to define a compact Hausdorff topology  $\mathcal{T}$  on  $X$ , by requiring a set  $U \subset X$  to be open if  $\Phi(U)$  is an open subset of  $\hat{X}$ .

<sup>4</sup>Here  $l \vee h$  stands for  $\max\{l, h\}$ .

<sup>5</sup>Whenever  $\mathcal{T}$  is a compact Hausdorff topology on  $\mathbb{N}$ ,  $(\mathbb{N}, \mathcal{T})$  is a Baire space as a complete metric space, hence it cannot be written as a countable union of rare sets, but every non isolated point  $n \in \mathbb{N}$  gives a rare singleton  $\{n\}$ . In particular, the set of natural numbers  $\mathbb{N}$  cannot be given of a connected compact Hausdorff topology; anyway a connected Hausdorff topology on  $\mathbb{N}$  is available: for instance *Golomb* topology, see [3].

be sought as a closed subspace of the dual space  $X^*$ , which is 1-*norming*<sup>6</sup> and *norm-attaining*, namely each linear functional belonging to the subspace is required to attain its norm on the unit ball of  $X$ .

When  $X$  is a separable conjugate space, the conditions above are also sufficient for a closed subspace  $M \subset X^*$  to be canonically a predual of  $X$  as it is shown in [5].

Here canonically means that the isometric isomorphism  $X \cong M^*$  is nothing but the restriction of the canonical injection  $j : X \rightarrow X^{**}$  to  $M$ .

Before stating the result announced, let us fix some notations:  $e \in l^\infty$  is the sequence constantly equal to 1,  $M_+$  stands for the positive<sup>7</sup> cone of a subspace  $M \subset l^\infty$ , while  $a^{\frac{1}{2}}$  is the square root<sup>8</sup> of a positive element  $a \in l_+^\infty$ .

According to the next theorem the spaces  $C_\tau(\mathbb{N})$  are precisely those  $l_1$ -predual rich of positive elements:

**Theorem 6.** *Let  $M \subset l^\infty$  be a predual of  $l_1$ , such that:*

- (a)  $e \in M$ .
- (b)  $M_+$  is weakly\*-dense in  $l_+^\infty$ .
- (c) If  $x \in M_+$ , then  $x^{\frac{1}{2}} \in M_+$ .

*Then  $M = C_\tau(\mathbb{N})$  for a suitable compact Hausdorff topology on the set of natural numbers  $\mathbb{N}$ .*

*Proof.* Let be  $\mathfrak{A} \subset l^\infty$  be the unital  $C^*$ -algebra<sup>9</sup> generated by  $M$ . If  $\omega$  is a *pure* (multiplicative) state on  $\mathfrak{A}$ , we can consider its restriction  $\omega|_M$ . Since  $M^* \cong l_1$ , we have  $\omega(x) = \varphi_y(x) \doteq \sum_i y_i x_i$  for each  $x \in M$ , where  $y$  is a suitable sequence in  $l_1$ . Now pick a positive element  $a \in l^\infty$ . Thanks to (b), there is a sequence  $\{x_n\}_{n \in \mathbb{N}} \subset M_+$  such that  $x_n \rightharpoonup a$  (in the weak\* topology of  $l^\infty$ ). Then we have

$$\begin{aligned} \varphi_y(a) &= \lim_n \varphi_y(x_n) = \lim_n \varphi \left( x_n^{\frac{1}{2}} x_n^{\frac{1}{2}} \right) = \\ \lim_n \omega \left( x_n^{\frac{1}{2}} x_n^{\frac{1}{2}} \right) &= \lim_n \omega \left( x_n^{\frac{1}{2}} \right) \omega \left( x_n^{\frac{1}{2}} \right) = \varphi_y(a^{\frac{1}{2}})^2 \end{aligned}$$

where the last equality holds since  $x_n^{\frac{1}{2}} \rightharpoonup a^{\frac{1}{2}}$  (the weak\* convergence in  $l^\infty$  is nothing but the bounded pointwise convergence).

If  $e_i \in l^\infty$  is the sequence given by  $e_i(k) = \delta_{i,k}$ , we get  $\varphi_y(e_i) = \varphi_y(e_i)^2$ , because  $e_i^{\frac{1}{2}}$  is  $e_i$  itself. It follows that, for each  $i \in \mathbb{N}$ ,  $\varphi_y(e_i)$  is 0 or 1. Since  $\sum_i |y_i| = \|\varphi_y\| = 1$ , one has  $y = e_k$  for some  $k$ . It easily follows that  $\omega$  is the evaluation map at  $k$ .

This means that  $\sigma(\mathfrak{A}) \cong \mathbb{N}$ , hence  $\mathfrak{A} = C_\tau(\mathbb{N})$ ,  $\mathcal{T}$  being the weak\* topology on the spectrum of  $\mathfrak{A}$ .

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<sup>6</sup>A subspace  $M \subset X^*$  is said to be 1-norming if for each  $x \in X$ , one has

$$\|x\| = \sup\{|\varphi(x)| : \varphi \in M_1\}$$

$M_1$  being the unit ball of  $M$ .

<sup>7</sup>An element  $x \in l^\infty$  is said to be positive if  $x_i \geq 0$  for each  $i \in \mathbb{N}$ ; in this case one writes  $x \geq 0$ .

<sup>8</sup>If  $x \geq 0$ , then  $x^{\frac{1}{2}}$  is the positive sequence given by  $x^{\frac{1}{2}}(i) \doteq x_i^{\frac{1}{2}}$  for each  $i \in \mathbb{N}$ .

<sup>9</sup>For a basic treatment of  $C^*$ -algebras theory, we refer the reader to [2].

Thanks to proposition 1, we have  $C_\tau(\mathbb{N}) \cong l_1$ ; since no proper inclusion relationships are allowed between preduals, we finally get  $M = \mathfrak{A}$ . This concludes the proof.  $\square$

## References

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